Application of Mohand Transform for Solving Linear Partial Integro – Differential Equations

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Abstract - Partial integro-differential equations (PIDE) occur naturally in various fields of science, engineering and social sciences. In this paper, we convert the proposed PIDE to an ordinary differential equation (ODE) using a Mohand transform (MT). Solving this ODE and applying inverse MT an exact solution of the problem is obtained.

Index Terms- Linear Partial integro-differential equations ; Mohand transform

1. INTRODUCTION

Real life phenomena are often modelled by ordinary/partial differential equations. Due to the local nature of ordinary differential operator(ODO), the models containing merely ODOs do not help in modelling memory and hereditary properties. One of the best remedies to overcome this drawback is the introduction of integral term in the model. The ordinary/partial differential equation along with the weighted integral of unknown function gives rise to an integro-differential equation (IDE) or a partial integro-differential equation (PIDE) respectively. Analysis of such equations can be found in [1-4].

Mohand transform method[5,6,7] is a useful tool for the solution of the response of differential and integral equation, and linear system of differential and integral equation.

The object of the present study is to determine exact solutions for linear partial integro- differential equations using Mohand transform without large computational work.

2. MOHAND TRANSFORM

Mohand transform of the function f (t) is defined as

$$M[f(t)] = R(v) = v^2 \int_0^\infty f(t)\, e^{-vt} dt, \ t \geq 0,$$
$$k_1 \leq v \leq k_2 \tag{1}$$

Theorem 2.1:

Mohand transform of partial derivatives are in the form:

$$M\left[\frac{\partial}{\partial t} f(x,t)\right] == vR(x,v) - v^2 f(x,0)$$
$$M\left[\frac{\partial^2 f}{\partial t^2}\right] = v^2 R(x,v) - v^3 f(x,0) - v^2 \frac{\partial}{\partial t} f(x,0)$$
$$M\left[\frac{\partial}{\partial x} f(x,t)\right] = \frac{d}{dx}[R(x,v)]$$
$$M\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{d^2}{dx^2}[R(x,v)]$$

Proof:

Mohand transform of the function $f(t)$ is defined as

$$M[f(t)] = R(v) = v^2 \int_0^\infty f(t)\, e^{-vt} dt, \ t \geq 0,$$
$$k_1 \leq v \leq k_2 \tag{2}$$

To obtain Mohand transform of partial derivatives , we use integration by parts as follows:

$$M\left[\frac{\partial}{\partial t} f(x,t)\right] = v^2 \int_0^\infty \frac{\partial}{\partial t} e^{-vt}\ dt$$

$$= \lim_{p \to \infty} \int_0^p v^2 e^{-vt} \frac{\partial f}{\partial t}\ dt$$

$$= \lim_{p \to \infty}\left[\left(v^2 e^{-vt} f(x,t)\right)_0^p - \int_0^p v^2 (-v) e^{-vt} f(x,t) dt\right]$$

$$= \left(v^2 e^{-vt} f(x,t)\right)_0^\infty + v^3 \int_0^\infty e^{-vt} f(x,t) dt$$
$$= - v^2 f(x,0) + vR(x,v)$$
$$\tag{3}$$

We assume that , f is piecewise continuous and is of exponential order.

Consider,

$$M\left[\frac{\partial}{\partial x} f(x,t)\right] = \int_0^\infty v^2 e^{-vt} \frac{\partial f(x,t)}{\partial x} dt$$

$$= \frac{\partial}{\partial x} \int_0^\infty v^2 e^{-vt} f(x,t) dt \quad \text{(using the Leibnitz' rule)}$$

$$= \frac{\partial}{\partial x}[R(x,v)] \quad \text{and}$$

$$M\left[\frac{\partial}{\partial x} f(x,t)\right] = \frac{d}{dx}[R(x,v)]$$
$$\tag{4}$$

Also we can find $M\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{d^2}{dx^2}[R(x,v)]$

$$\tag{5}$$

To find $\quad M\left[\frac{\partial^2 f}{\partial t^2}\right]$

Let $\frac{\partial f}{\partial t} = g$, *then*

From equation (2), we have

$$M\left[\frac{\partial^2 f}{\partial t^2}\right] = M\left[\frac{\partial g}{\partial t}\right]$$
$$= M[vg(x,t) - v^2 g(x,0)]$$
$$= vM[g(x,t)] - v^2 M[g(x,0)]$$

$$= v^2 R(x,v) - v^3 f(x,0) - v^2 \frac{\partial}{\partial t} f(x,0)$$
$$(6)$$

We can easily extend this result to the nth partial derivative by using mathematical induction.

Theorem 2.2 (Convolution)

Let $f(t)$ and $g(t)$ having, Mohand transforms $M(v)$ and $N(v)$, then Mohand transform of the convolution of f and g are

$$(f * g)(t) = \int_0^\infty f(t)g(t - \tau)\, d\tau \text{ is given by}$$

$$M[(f * g)(t)] = \frac{1}{v^2} M(v)N(v)$$

Now to illustrate the method we consider the general linear partial integro differential equation,

$$\sum_{i=0}^m a_i \frac{\partial^i u(x,t)}{\partial t^i} + \sum_{i=0}^n b_i \frac{\partial^i u(x,t)}{\partial x^i} + cu +$$
$$\sum_{i=0}^r d_i \int_0^t k_i(t - s) \frac{\partial^i u(x,s)}{\partial x^i} + f(x,t) = 0$$
$$(7)$$

Applying the Mohand Transform to both sides of (7), we have

$$\sum_{i=0}^m a_i M\left[\frac{\partial^i u(x,t)}{\partial t^i}\right] + \sum_{i=0}^n b_i M\left[\frac{\partial^i u(x,t)}{\partial x^i}\right] +$$
$$cM[u] + \sum_{i=0}^r d_i M\left[\int_0^t k_i(t - s) \frac{\partial^i u(x,s)}{\partial x^i}\right] +$$
$$M[f(x,t)] = 0$$
$$(8)$$

Using theorem 1 and theorem 2 for Mohand transform, we get

$$\sum_{i=0}^m a_i [v^i R(x,v) - v^{i+1} u(x,0) - v^i u_t(x,0) -$$
$$\cdots u_{ttt\ldots(i+1)times}(x,0)] + \sum_{i=0}^n b_i \frac{d^i}{dx^i} R(x,v) +$$
$$cR(x,v) + \sum_{i=0}^r d_i \bar{u}_i(v) \frac{d^i}{dx^i} R(x,v) + \bar{f}(x,v) = 0$$
$$(9)$$

Where
$$M[u(x,t)] = R(x,v), M[u_i(t)] =$$
$$\bar{u}_i(v) \text{ and } M[f(x,t)] = \bar{f}(x,v).$$

After using prescribed conditions, equation (9) represents an ordinary differential equation with dependent variable R(x,v). After solving this ordinary differential equation and taking inverse Mohand transform of R(x,v), we have the required solution u(x,t) of (7).

3. ILLUSTRATIVE EXAMPLES

In this section we solve first order partial-integro differential Equations and the Second order partial-integro differential equation, wave equation, heat equation, Laplace equation and Telegraphers equation which are known as four Fundamental equations in mathematical physics and occur in many branches of physics, in applied mathematics as well as in engineering.

Example 3. 1

Consider the linear partial integro- differential equation ,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial x} + 2 \int_0^t (t - s)u(x,s)\, ds - 2e^x$$
$$(10)$$

with initial condition u(x,0) = e^x , u_t (x,0) =0 and boundary condition u(0,t)=cost

Taking Mohand transform of Eq. (5), we have

$$M[u_{tt}] = M[u_x] + 2M\left[\int_0^t (t - s)\, u(x,s)ds\right] - 2e^x M[1]$$
$$(11)$$

$$v^2 R(x,v) - v^3 u(x,0) - v^2 u_t(x,0) = \frac{d}{dx} R(x,v)$$
$$+ \frac{2}{v^2} M[x] M[u(x,s)] - 2e^x M[1]$$

$$R'(x,v) + \left(\frac{2}{v^2} - v^2\right) R(x,v) = (2v - v^3)e^x$$
$$(12)$$

$$R(x,v) = \frac{v^3}{v^2+1} e^x + c e^{\left(v^2 - \frac{2}{v^2}\right)x}$$
$$(13)$$

Now , $M[u(0,t)] = R(0,v) = M[cost] = \frac{v^3}{v^2+1}$
$$(14)$$

Compare (13) and (14) , we get c = 0

Equation (13) implies ,

$R(x,v) = e^x \frac{v^3}{v^2+1}$
$$(15)$$

Applying inverse Mohand transform on both sides,

u(x,t) = M^{-1}[R(x,v)]
$= e^x cost$

Example 3. 2

Consider the linear partial integro- differential equation ,

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$$u_t - u_{xx} + u + \int_0^t e^{t-s} \, u(x,s) \, ds = (x^2+1)e^t - 2$$

(16)

$$u(x,0) = x^2, \quad u_t(x,0) = 1 \tag{17}$$

$$u(0,t) = t, \quad u_x(0,t) = 0 \tag{18}$$

Taking Mohand transform of Eq. (16), we have

$$M[u_t] - M[u_{xx}] + M[u] +$$
$$M\left[\int_0^t e^{t-s} \, u(x,s) \, ds\right] = (x^2+1)M[e^t] - 2M[1]$$

(19)

$$vR(x,v) - v^2 u(x,0) - \frac{d^2 R}{dx^2} + R + \frac{v^2 R}{(v-1)v^2} =$$
$$(x^2+1)\frac{v^2}{v-1} - 2v$$

$$\frac{d^2 R}{dx^2} - R\left[\frac{v^2}{v-1}\right] = \frac{-v^3}{v-1}x^2 + \frac{v^2-2v}{v-1} \tag{20}$$

Solving (20) we get,

$$R(x,v) = c_1 e^{\frac{v}{\sqrt{v-1}}x} + c_2 e^{-\frac{v}{\sqrt{v-1}}x} + x^2 v + 1$$

(21)

Now, $u(0,t) = t,$ so

$$R(0,v) = M[u(0,t)] = M[t] = 1$$

(22)

Using (21) and (22), we get

$$c_1 + c_2 = 0$$

(23)

And $c_1 - c_2 = 0$

(24)

Solving (23) and (24), we get $c_1 = c_2 = 0.$
Then equation (16) becomes

$$R(x,v) = x^2 v + 1$$

(25)

Applying inverse Mohand transform on both sides,

$$u(x,t) = x^2 M^{-1}[v] + M^{-1}[1]$$
$$= x^2 + t$$

4. CONCLUSION
In this paper, we have successfully developed the Mohand transform for solving linear partial integro-differential equation. The given application shows that the exact solution have been obtained using very less computational work and spending a very little time.

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